For the satellite below Earth,

$$A = 1/\beta \tag{23}$$

Only the first solution is physically meaningful due to Eq. (7). In fact, for all ground points within the field of view of the satellite, it must be true that $1 \geq A \geq \beta$. When the second factor of Eq. (20) is examined, its maximum value corresponds to the maximum angular rate between the ground and satellite vectors:

$$(\dot{\theta}/\omega)^2 = (\omega_R/\omega)^2 \tag{24}$$

where ω_R is the relative angular rate.

The relative angular velocity of any ground point relative to the satellite is given by

$$\boldsymbol{\omega}_R = \boldsymbol{\omega}_G - \boldsymbol{\omega}_S \tag{25}$$

Now examine the relative angular rate,

$$\omega_R = |\boldsymbol{\omega}_R| = \sqrt{|\boldsymbol{\omega}_G|^2 + |\boldsymbol{\omega}_S|^2 - 2\boldsymbol{\omega}_G \cdot \boldsymbol{\omega}_S} = \omega\sqrt{2[1 - \cos(i)]} \tag{26}$$

Therefore

$$(\dot{\theta}/\omega)^2 = (\omega_R/\omega)^2 = 2[1 - \cos(i)] \tag{27}$$

Substituting Eqs. (22) and (27) in Eq. (20) yields

$$C_{\max} = 2\beta[1 - \cos(i)] \tag{28}$$

Thus, the maximum range rate among all ground points within the field of view of the satellite becomes

$$\dot{r}_{\max} = \omega\sqrt{r_S r_G C_{\max}} = r_G\omega\sqrt{2[1 - \cos(i)]} \cong r_G\omega\sin(i) \qquad \text{for} \quad i \ll 1 \tag{29}$$

We can also derive the preceding maximum range rate results with another approach that is less mathematically rigorous, yet more physically insightful, as follows.

The Cartesian position vector $\boldsymbol{R}_G$ of any ground point depends on time, its longitude, and its latitude. It is not a simple function. However, if we look at the kinematics of all ground points from the angular velocity perspective, it is actually very simple. All ground points have the same instantaneous angular speed because they are part of one single rigid body. Yet the axis of this angular velocity oscillates from the satellite's point of view due to its motion.

From Fig. 1, we can see that all ground points within the field of view of the satellite are moving at the angular rate ω_R. There is only one possible point P moving away from the satellite at the maximum speed or range rate $r_G\omega_R$. Therefore, we conclude that

$$\dot{r}_{\max} = r_G\omega_R = r_G\omega\sqrt{2[1 - \cos(i)]} \tag{30}$$

which is the same result as Eq. (29) derived earlier. For the example of a GEO satellite orbit with inclination 5.3 deg, the maximum possible range rate is given by $\dot{r}_{\max} = 0.043$ km/s.

Conclusions

We produced a closed-form algorithm and useful equation that permits easy calculation of the range, range rate, and range acceleration for the case of an inclined GEO satellite. We showed two methods to achieve the same results for finding the maximum range rate.

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Variational Calculus and Approximate Solutions of Algebraic Equations

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Introduction

THE thrust of this Note is the derivation of the equations that are to be solved for an approximate analytical solution of an algebraic equation involving a small parameter, the algebraic perturbation problem. The usual procedure is to recognize that the solution can be expressed in a power series of the small parameter, substitute the series into the algebraic equation, expand in a Taylor series, and equate the coefficients of the powers of the small parameter to zero. This process leads to the equations for the zeroth-order part of the solution, the first-order part, and, so on.

In association with parameter optimization,[1] it has been shown that the Taylor series process is equivalent to a differential process called variational calculus. A variation is a small but finite displacement, its symbol behaves like a differential, and variations of independent variations are zero whereas variations of dependent variations are not zero. Variational calculus has been applied to the initial value problem of ordinary differential equations.[2] Perturbed paths have been generated by perturbing the initial conditions or a small parameter in the differential equation.

In this Note, variational calculus is established for algebraic equations. The expansion process and the variational process are both used to derive the equations for the zeroth-order part, the first-order part, etc., of the algebraic perturbation problem of Kepler's equation with small eccentricity. The efforts required by both processes are compared.

Although only a scalar equation is considered here, the variational process can be applied to vector equations if indicial notion is employed. Matrix notation can be employed if only the first-order correction to the zeroth-order solution is being sought.

Taylor Series Expansions and Variational Calculus

In this section it is shown that Taylor series expansions can be created on a term by term basis by a differential process called variational calculus. To do so, consider the scalar equation

$$y = F(x) \tag{1}$$

which is shown schematically in Fig. 1. In this formulation, x is the independent variable, and y is the dependent variable. Consider the nominal point x, y and the perturbed or neighboring point x_*, y_*. If the symbol Δ denotes a small but finite change, a displacement, the coordinates of the perturbed point are $x_* = x + \Delta x$ and $y_* = y + \Delta y$. For a given Δx, it is desired to find Δy. From $y_* = F(x_*)$, it is seen that

$$y + \Delta y = F(x + \Delta x) \tag{2}$$

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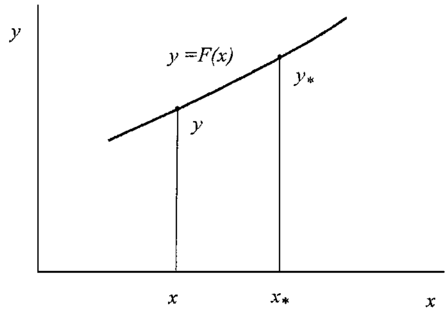

Fig. 1 Neighboring points.

so that a Taylor series expansion about x combined with $y = F(x)$ leads to

$$\Delta y = F_x(x)\Delta x + \tfrac{1}{2!}F_{xx}(x)\Delta x^2 + \cdots \tag{3}$$

Here, the subscript x denotes a partial derivative with respect to the general argument of the function $F(x)$. On the other hand, the argument x in Eq. (3) denotes the point x about which the expansion is made.

Note that Δy has a first-order part in Δx, a second-order part in Δx, and so on. Whereas the quantity Δy changes with x, the quantity Δx is given and does not change with x. Hence, it only has a first-order part. As a consequence, Δy and Δx are written as

$$\Delta y = \delta y + \tfrac{1}{2!}\delta^2 y + \cdots, \qquad \Delta x = \delta x \tag{4}$$

where δ is a first-order quantity, δ^2 a second-order quantity, and so on. Comparing Eqs. (3) and (4) shows that

$$\delta y = F_x(x)\delta x \tag{5}$$

$$\delta^2 y = F_{xx}(x)\delta x^2$$

$$\vdots \tag{6}$$

At this point, it is observed that δ, called a variation, appears to behave like a differential. To show that it does, taking the variation of $y = F(x)$ leads to

$$\delta y = F_x(x)\delta x \tag{7}$$

Then, taking the variation of the first variation gives

$$\delta^2 y = F_{xx}(x)\delta x^2 + F_x(x)\delta^2 x \tag{8}$$

However, δx is constant, that is, it is not a function of x, so that $\delta(\delta x) = 0$, and the second variation becomes

$$\delta^2 y = F_{xx}(x)\delta x^2 \tag{9}$$

Thus far, the argument x is the general argument of the function $F(x)$. If Eqs. (7) and (9) are evaluated at the point x of Fig. 1, they become identical with Eqs. (5) and (6). Hence, the variation δ behaves like a differential. In general, variations of independent variations are zero, but variations of dependent variations are not, that is,

$$\delta(\delta x) = 0, \qquad \delta(\delta y) = \delta^2 y \tag{10}$$

In conclusion, each term of a Taylor series can be obtained by taking the variation of the general form of the previous term.

Algebraic Perturbation Problem

The algebraic perturbation problem is to find an approximate analytical solution of the scalar equation

$$f(z, \varepsilon) = 0 \tag{11}$$

where the scalar z is the unknown and ε is a given, small, scalar parameter. Hence, ε is the independent variable, and z is the dependent variable. It is assumed that an analytical solution exists for the equation $f(z, 0) = 0$; otherwise, there is no reason to continue. In general, the solution of Eq. (11) has the functional form

$$z = g(\varepsilon) \tag{12}$$

For small ε, a Taylor series expansion leads to

$$z = g(0) + g_\varepsilon(0)\varepsilon + \tfrac{1}{2!}g_{\varepsilon\varepsilon}(0)\varepsilon^2 + \cdots \tag{13}$$

meaning that the solution can be expressed as a power series in ε.

In what follows, the equations for finding an approximate analytical solution of an algebraic equation can be obtained by applying the expansion and variational processes to the general problem (11). In practice, however, the equations for the approximate analytical solution of an equation would probably be derived by working with the particular equation directly because many of the derivatives in the general approach would be zero.

The particular problem used here to demonstrate these techniques is the approximate solution of Kepler's equation[3] with small eccentricity, that is,

$$E - e\sin E - M = 0 \tag{14}$$

where E is the eccentric anomaly of a point in an orbit, M is the known mean anomaly of the point, and e is the known eccentricity of the orbit. The orbit is assumed to be nearly circular so that the eccentricity e is small. In Kepler's equation, e is the independent variable, and E is the dependent variable.

Expansion Process for Kepler's Equation

To derive the equations for an approximate analytical solution of Kepler's equation (14) for a given value of e by the expansion process, the solution E_* is expressed in the form of Eq. (13) as

$$E_* = E_0 + E_1 e + E_2 e^2 + \cdots \tag{15}$$

and it is desired to find the equations for the zeroth-order solution E_0, the first-order solution E_1, and so on. To begin, E_* is rewritten as

$$E_* = E_0 + \Delta E \tag{16}$$

where

$$\Delta E = E_1 e + E_2 e^2 + \cdots \tag{17}$$

From Eq. (16), it is seen that

$$\sin E_* = \sin E_0 \cos\Delta E + \cos E_0 \sin\Delta E \tag{18}$$

Then, with the expansions for sine and cosine, Kepler's equation becomes

$$E_0 + \Delta E - e\big[\sin E_0(1 - \Delta E^2/2! + \cdots) + \cos E_0(\Delta E - \Delta E^3/3! + \cdots)\big] - M = 0 \tag{19}$$

Finally, substitute Eq. (17) into Eq. (19), rearrange terms in ascending powers of e, and set the coefficients of the various-order terms to zero to obtain the following results:

$$E_0 = M$$

$$E_1 = \sin E_0$$

$$E_2 = -E_1 \cos E_0$$

$$\vdots \tag{20}$$

Once the solutions of these equations for E_0, E_1, E_2, ... have been obtained, the approximate analytical solution of Kepler's equation is given by Eq. (15).

It should be obvious that the work required to add another order is significant. All of the mathematical operations in the equation must be redone with the new series.

Variational Process for Kepler's Equation

To derive the equations for an approximate analytical solution of Kepler's equation by the variational process, the solution (13) is expressed as

$$E_* = E + \delta E + \tfrac{1}{2!}\delta^2 E + \cdots \tag{21}$$

and it is desired to find the equations for the zeroth-order solution E, the first-order solution δE, and so on. Taking variations of Kepler's equation (14) leads to

$$\begin{aligned} E - e\sin E - M &= 0 \\ \delta E - \sin E\delta e + e\cos E\,\delta E &= 0 \\ \delta^2 E + 2\cos E\,\delta e\delta E - e\sin E\,\delta E^2 + e\cos E\,\delta^2 E &= 0 \\ &\vdots \end{aligned} \tag{22}$$

where $\delta(\delta e) = 0$ because δe is an independent variation. Then, evaluating the coefficients at $e = 0$ (E becomes the zeroth-order solution) gives the following:

$$\begin{aligned} E &= M \\ \delta E &= \sin E\,\delta e \\ \delta^2 E &= -2\cos E\,\delta e\delta E \\ &\vdots \end{aligned} \tag{23}$$

Once Eqs. (23) have been solved for $E, \delta E, \delta^2 E, \ldots$, the solution of Kepler's equation is given by Eq. (21).

That these equations are the same as Eqs. (20) can be shown by making the connections

$$\delta e = e, \qquad E = E_0, \qquad \delta E = E_1 e$$
$$\tfrac{1}{2!}\delta^2 E = E_2 e^2, \qquad \cdots \tag{24}$$

The first equation comes from the fact that $\Delta e = \delta e$ because δe is an independent variation, that is, $\delta(\delta e) = 0$. Also, the total change in e is the value of e on the perturbed path minus the value of e on the nominal path, that is, $\Delta e = e - 0 = e$. The remaining equations come from comparing Eqs. (15) and (21).

To add another order is easy; only differentiation is needed. Admittedly, the complexity of each variation increases with order, but some of it goes away if terms can be combined.

Conclusions

Variational calculus has been developed for algebraic equations. It has been shown that Taylor series expansions can be made on a term by term basis by applying a differential (variational) process wherein variations of independent variations are zero and variations of dependent variations are not zero.

To establish their relative merits, the expansion process and the variational process have been applied to the algebraic perturbation problem in the form of Kepler's equation with small eccentricity. The variational process is clearly superior to the expansion process. In the expansion process, all of the mathematical operations must be carried out in terms of series, so that it is laborious to derive the equations for another-order solution. With the variational process, only differentiation is required, and the equation for another order can be obtained easily from the general form of the previous-order equation.

Usually, it is difficult to ensure that the various-order equations are correct regardless of the method being used. Having two ways to derive the equations helps alleviate this problem.

Although only a scalar equation is considered here, variational calculus can be applied to vector equations by employing indicial notation. If only the first-order correction to the zeroth-order solution is being sought, matrix notation can be employed.

Whereas the algebraic perturbation problem has been used to demonstrate the benefits of variational calculus, any problem involving a Taylor series expansion could have been used.

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Near Minimum-Time Trajectories for Solar Sails

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I. Introduction

SOLAR sailing has long been considered for a diverse range of future mission applications.[1] As with other forms of low-thrust propulsion, trajectory optimization has been a focus of development activities. In particular, minimum-time solar-sail trajectories have been obtained by several authors for a range of mission applications. Almost all of these studies have used the Pontryagin principle of the calculus of variations to obtain minimum-time trajectories by the classical, indirect method (see, for example, Ref. 2). The indirect approach provides a continuous time history for the required solar-sail steering angles. Only a few studies have used the competing direct approach, which recasts the task as a parameter optimization problem by discretizing the control variables. These studies have used many discrete segments for the sail steering angles to ensure a close approximation to the continuous steering angles provided by the indirect approach and hence a close approximation to the true minimum-time trajectory (see, for example, Ref. 3).

In this Note it is demonstrated that near minimum-time solar-sail trajectories can be obtained from the direct method using relatively few discrete segments. For trajectories involving transfers between near circular orbits, as few as three segments will yield a trajectory that satisfies the two-point boundary conditions of the trajectory while minimizing the transfer time to within a few percent of the absolute minimum transfer time. The motivation for this investigation arose from the observation that determination of minimum-time solar-sail trajectories is difficult because "the performance index is extremely insensitive to small variations around the optimal sail steering history."[4] Because the performance index is rather flat, it is expected that simple steering laws, which are operationally efficient to implement, will also be close to minimum time.

II. Fixed-Attitude Steering

To solve the minimum-time problem using direct methods, the sail steering angles are defined a priori in segments using a set of free parameters. Typically, the steering angles are parameterized in each segment as line elements or cubic splines to provide a continuous

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